λ Fuzzy Measure Identification Methods using λ and Weights

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1 Introduction

This document is the English summary of the paper[1] to shows the fuzzy measure identification methods using λ and those algorithms.

2 λ Fuzzy Measure

Fuzzy integrals are useful tool for global evaluation models. However, the number of parameter of fuzzy measure is large. λ fuzzy measure is one class of fuzzy measures which can be identified by interaction index λ (or ξ) and weights of individual evaluation items(Figure 1).

\[ \mu_{\lambda}(A \cup B) = \mu_{\lambda}(A) + \mu_{\lambda}(B) + \lambda \mu_{\lambda}(A)\mu_{\lambda}(B) \] (1)

Figure 1: λ Fuzzy Measure and Fuzzy Integral

For a \( \lambda > -1 \), λ fuzzy measure \( \mu_{\lambda} \) is defined as:
for all $A \cap B = \emptyset$ where $\mu(X) = 1$ and $\mu(A) \in [0, 1], \forall A \in 2^X$. However, if the $\lambda$ is fixed, the $\mu$ is not identified uniquely. To identify the fuzzy measure uniquely, we must also specify the weights $(w_1, \ldots, w_n, w_i \geq 0)$ and identification standard.

The weights of a fuzzy measure are not trivial. Therefore, 3 standards are introduced.

- Singleton Fuzzy Measure Ratio Standard
- Shapley Value Standard
- Input Number Standard

3 Singleton Fuzzy Measure Ratio Standard

This standard is to identify the fuzzy measure such that:

$$\mu_{\lambda}({\{1\}}) : \mu_{\lambda}({\{2\}}) : \ldots : \mu_{\lambda}({\{n\}}) = w_1 : w_2 : \ldots : w_n$$

(2)

Therefore, this standard makes point of each input’s single influence to the output.

Identification Algorism

Varying $p \in (0, 1)$ and find $\mu({\{1, 2, \ldots, n\}}) = 1$.

1. Normalise weights where $\max w_i = 1$.
2. $p := 0.5$
3. $\mu({\{i\}}) := pw_i, \forall i$.
4. Calculate
   $$\mu({\{1, 2, \ldots, j\}}) := \mu({\{1, 2, \ldots, j-1\}}) + \mu({\{j\}}) + \lambda \mu({\{1, 2, \ldots, j-1\}})\mu({\{j\}})$$
   for $j = 2, \ldots, n$
5. if $\mu({\{1, 2, \ldots, j\}}) > 1$ for a $j$ then decrease the $p$ and go to 3.
6. if $\mu({\{1, 2, \ldots, n\}}) < 1$ then increase the $p$ and go to 3.
7. if $\mu({\{1, 2, \ldots, n\}}) = 1$ then stop the algorism.

4 Shapley Value Standard

In this standard, weights are positive values. This standard is to identify the fuzzy measure $\mu_{\lambda}$ such that:

$$sh_i(\mu_{\lambda}) = w_i, \forall i$$

(3)

where $sh_i(\mu_{\lambda})$ is the Shapley Value of $i$-th evaluation item of the fuzzy measure $\mu_{\lambda}$. Shapley Value is defined as:

$$sh_i(\mu) = \sum_{S \subseteq X} \gamma_n(S)[\mu(S) - \mu(S \setminus \{i\})]$$

(4)

$$\gamma_n(S) = \frac{(n - |S|)!(|S| - 1)!}{n!}$$

(5)

This standard make point of each input’s weight.
Identification Algorithm

1. Normalise weights where $\sum w_i = 1$.
2. $q_i := 1, \forall i$ (initial)
3. Identify the fuzzy measure $\mu_\lambda$ using Singleton Fuzzy Measure Ratio Standard where $q_1 : q_2 : \ldots : q_n = \mu_\lambda(\{1\}) : \mu_\lambda(\{2\}) : \ldots : \mu_\lambda(\{n\})$.
4. Calculate Shapley values $sh_i(\mu_\lambda), \forall i$.
5. If $sh_i(\mu_\lambda) = w_i, \forall i$ then stop the algorithm.
6. Find $k$ where $w_k - sh_k(\mu_\lambda) = \max_i[w_i - sh_i(\mu_\lambda)]$.
7. $q_k := q_k + 1$
8. go to 3.

5 Input Number Standard

In this standard, weights are non-negative integers. Weights are the numbers of the same weights inputs. For example, if $n = 3$ and $w_1 = 2, w_2 = 4$ and $w_3 = 1$ then there are 7 same weights inputs. That is, 2 inputs are $h(1)$, 4 are $h(2)$ and one is $h(3)$ (see Figure 2).

![Figure 2: Input Number Standard](image)

Identification Using $\phi_s$ transformation, it is easy to identify the $\lambda$ fuzzy measure. $u_i$ are normalized weights where $u_i = w_i / \sum_i w_i$.

$$\phi_s : [0, 1] \to [0, 1], \ s \in [0, +\infty] \quad (6)$$
\( \phi_s(u) = \begin{cases} 
(u) & \text{if } s = 0 \\
 u & \text{if } s = 1 \\
 1 - (1 - u) & \text{if } s = +\infty \\
 (su - 1)/(s - 1) & \text{otherwise} 
\end{cases} \)  \hspace{1cm} (7)

where \( (u) = \begin{cases} 
1 & \text{if } 0 < u \leq 1 \\
0 & \text{if } u = 0 
\end{cases} \)

where \( s = \lambda + 1 \).

\[ \mu_\lambda(A) = \phi_{\lambda+1}(\sum_{i \in A} u_i) \]  \hspace{1cm} (8)

For example, if \( \lambda = 2 \) then \( \mu_{\lambda}([1, 3]) = \phi_{\lambda+1}([w_1 + w_3]) = \phi_{3/7} = \frac{3/7 - 1}{3 - 1} = 0.30 \).

\( \xi \)  \( \xi \) is another interaction index which values \([0, 1]\).

\[ \xi : [0, \infty] \to [0, 1] \]

\[ \xi(s) = \begin{cases} 
1 & \text{if } s = 0 \\
0 & \text{if } s = +\infty \\
\frac{1}{1 + \sqrt{s}} & \text{otherwise} 
\end{cases} \]  \hspace{1cm} (9)

\[ s = \xi^{-1}(x) = \begin{cases} 
+\infty & \text{if } x = 0 \\
\frac{1-x}{x^2} & \text{if } x \neq 0 
\end{cases} \]  \hspace{1cm} (10)

\( \xi \in (0, 1) \) has one to one correspondence with \( \lambda \in (-1, \infty) \)(see fig.3).

Figure 3: \( \xi \) and \( \lambda \)

If \( \xi = 1 \) then the output value of the Choquet integral is the maximum value of inputs and if \( \xi = 0 \) then the output value is minimum. Therefore, the input number standard makes a point of interaction index.

6  Comparison

Figure 4 is the comparison among the 3 standards. Only input number standard is an extension of maximum and minimum outputs.
Output of Choquet Integral

$h(1)=10, h(2)=20, h(3)=30$

$w_1=2, w_2=4, w_3=1$

Input Number

Shapley Value

Singleton Fuzzy Measure Ratio

Figure 4: Comparison

References
