Properties of linear differential equations

For Classical Mechanics A, Univ.Tokyo (2017)

Descriptions: Reply to the question associated with the ansatz for the damped harmonic oscillator: why do we need to consider the linear combination?

[*Apr. 29, 2017*]

1 Damped harmonic oscillator

The equation of motion for the damped harmonic oscillator is given as

$$\left(m\frac{d^2}{dt^2} + 2m\gamma\frac{d}{dt} + m\omega^2\right)x(t) = 0. \tag{1}$$

As in Feynman Ch. 25, let us denote the differential operator in this equation as $\hat{L}$. Then, the equation is expressed

$$\hat{L}(x) = 0. \tag{2}$$

In order to solve the differential equation, three approaches were attempted in the previous lectures: trigonometric functions, the McLaurin (power) series expansion, and the ansatz method.

The last one, the ansatz method, is the method that we studied most recently, and it assumes a possible mathematical form of solutions with parameters to be determined. In the present case, the following ansatz is known to work

$$x(t) = e^{\lambda t}, \tag{3}$$

where λ is a complex parameter to be determined. Inserting this ansatz to the differential equation, we have a quadratic equation for λ

$$\lambda^2 + 2\gamma\lambda + \omega^2 = 0. \tag{4}$$

With the quadratic formula, we obtain

$$\lambda = \gamma \pm \sqrt{\gamma^2 - \omega^2}. \tag{5}$$

Let us classify this solution by an integer k in the following way

$$\lambda_k = \gamma + (-1)^k\sqrt{\gamma^2 - \omega^2}, \tag{6}$$

where $k = 0, 1$. Therefore, we have two possible solutions for the damped harmonic oscillator, which are

$$x_0(t) = e^{\lambda_0 t} \quad \text{and} \quad x_1(t) = e^{\lambda_1 t} \tag{7}$$

Obviously,

$$\hat{L}(x_k) = 0 \tag{8}$$

is satisfied for $k = 0, 1$.

In my previous lecture (on Wednesday, April 26, 2017), I said that the linear combination of the above two solutions $x_k(t)$

$$x(t) = c_0 x_0(t) + c_1 x_1(t), \quad c_k \text{ being a constant for } k = 0, 1 \tag{9}$$

is regarded as the "general" solution of the damped harmonic oscillator. Then, there was a question raised on this remark:

"You said that the ansatz is assumed to have a single-term form of the exponential, but you also said that the "general" solution is a linear combination of the two independent solutions, but it consists of two terms. This result looks contradicting to what you assumed in the beginning, doesn't it? We should rather employ the independent solutions separately, shouldn't we? Say, $x(t) = x_0(t)$, or $x(t) = x_1(t)$. What about this?"

This question really hits the centre of the so-called "linear problem" of the differential equation, which is also discussed in Feynman Ch. 25. Let us rebuild what is discussed in Feynman's textbook.

2 Linear problem

Differential operators $\frac{d^n}{dt^n}$ satisfies the following property

$$\frac{d^n}{dt^n}\left(af(t) + bg(t)\right) = a\frac{d^n}{dt^n}f(t) + b\frac{d^n}{dt^n}g(t), \tag{10}$$

where a, b are constants. This result means that the derivative of the linear combination of functions is equivalent to the linear combination of the derivatives of the individual functions. Therefore, if $x_0(t)$ and $x_1(t)$ are the two independent solutions of the equation of motion (1), then the linear combination of these independent solutions

$$x(t) = c_0 x_0(t) + c_1 x_1(t) \tag{11}$$

is another solution of Eq.(1). The proof of this remark relies on the "linear property" of Eq.(10):

> Suppose $\hat{L}(x_0) = 0$ and $\hat{L}(x_1) = 0$. Then,
>
> $$\hat{L}(c_0 x_0 + c_1 x_1) = c_0 \hat{L}(x_0) + c_1 \hat{L}(x_1) = 0. \tag{12}$$
>
> The above equation is justified thanks to the linear form of the differential equation (1).

In this way, we have millions of solutions to satisfy the equation of motion (1), by changing c_k. Such individual and special cases can be summarised into the general form (11). For example, $x(t) = e^{\lambda_0 t}$ simply corresponds to $(c_0, c_1) = (1, 0)$.

Furthermore, suppose that we can find more than three solutions to satisfy the equation of motion (1), say $\hat{L}(x_0) = 0$, $\hat{L}(x_1) = 0$, $\hat{L}(x_2) = 0$, then the linear combination of these three is again the solution of the equation of motion. However, the "degrees of freedom" of a differential equation of order n (i.e., a differential equation containing a term d^n/dt^n) is n, so that the equation of motion allows only two independent solutions (the equation of motion is a second-order differential equation). When x_0 and x_1 are such independent solutions, then x_2 can be expressed as a linear combination

$$x_2(t) = d_0 x_0(t) + d_1 x_1(t). \tag{13}$$

Hence,

$$\begin{aligned} x(t) &= c_0 x_0(t) + c_1 x_1(t) + c_2 x_2(t) \\ &= (c_0 + d_0)\, x_0(t) + (c_1 + d_1)\, x_1(t) \\ &= \mathcal{C}_0 x_0(t) + \mathcal{C}_1 x_1(t). \end{aligned} \tag{14}$$

The linear property of the equation of motion can be broken down, for instance, if we replace the linear frictional force $-2\gamma v(t)$ with the quadratic friction $-2\gamma' v(t)^2$, or if we replace the harmonic term $-\omega^2 x(t)$ with an anharmonic force $-\omega^2 \left(x(t) + \epsilon x(t)^3\right)$. Such equations of motion, or differential equations that contain "non-linear terms" are called "non-linear equations", in which the superposition principle, i.e., Eqs. (11) and (12), cannot be applied.

In fundamental theories of physics, linear equations are everywhere. For instance, the Maxwell equations in the classical electromagnetism, which is to be lectured in the A-semester, are linear differential equations. The Schrödinger equation in Quantum mechanics is also a linear equation. Therefore, the superposition principle can be applied to the electrodynamics and quantum mechanics, which plays a significant role to formulate the theories. Students are strongly recommended to confirm that the Maxwell and Schrödinger equations satisfy the linear property. (For instance, suppose Φ_0 and Φ_1 are two independent wave functions of the Schrödinger equation. Then, you should examine whether a linear combination $c_0\Phi_0 + c_1\Phi_1$ satisfies the same Schrödinger equation or not.) The concept of the superposition principle leads to an application of the Fourier series expansion and/or the Fourier transform to these linear problems. For instance, the electromagnetic field (or the vector potential) can be decomposed in accordance with the Fourier transform.

3 Practical problems if sticking with single-term solution

Let us now discuss what is not written in Feynman's textbook. Suppose that the single-term solution, $x_0(t)$ or $x_1(t)$, is employed rather than the general solution of the linear combination form. Then, what is wrong with it?

Firstly, we write down a solution, for instance, as

$$x(t) = e^{\lambda_0 t}. \tag{15}$$

Then, what is wrong with this? For the sake of simplicity, we consider the case $\gamma = 0$ in the below. Then the "solution" turns to be

$$x(t) = e^{i\omega t}. \tag{16}$$

As explained in the lecture, the position of a particle of interest needs to be described with a real dynamical variable. The above quantity is a complex variable, not a real one. You cannot measure a position of particles with complex numbers.

By utilising a freedom to multiply a complex number $C = a + ib$ to the solution, we might be able to transform the single-term solution to a real variable. Here, a and b are constant real numbers.

$$\begin{aligned} Ce^{i\omega t} &= (a + ib)\left(\cos(\omega t) + i\sin(\omega t)\right) \\ &= \{a\cos(\omega t) - b\sin(\omega t)\} + i\{b\cos(\omega t) + a\sin(\omega t)\} \end{aligned} \tag{17}$$

To make the imaginary part vanish, we require

$$b\cos(\omega t) + a\sin(\omega t) = 0, \tag{18}$$

but we know that these trigonometric functions are linearly independent to each other. The above condition is thus satisfied only when $a = b = 0$, which means $C = 0$. Surely, $x(t) = 0$ is the trivial solution, but does not carry any physical information with regard to the dynamics in Eq.(1). We have to conclude that the choice of a single-term solution ends up in an absurd result.

Next, we should examine whether or not the linear combination of Eq.(11) satisfies the physical requirement. To demand $x(t)$ to be a real variable, we should have

$$x(t) = x(t)^*. \tag{19}$$

In other words, the complex conjugate of the position variable must be equal to the original. With the superposition, we have

$$c_0 e^{i\omega t} + c_1 e^{-i\omega t} = c_0^* e^{-i\omega t} + c_1^* e^{i\omega t}, \tag{20}$$

or

$$(c_0 - c_1^*)e^{i\omega t} + (c_1 - c_0^*)e^{-i\omega t} = 0. \tag{21}$$

Exponential functions e^{ix} and e^{-ix} are linearly independent since their Wronskian

$$\begin{vmatrix} e^{ix} & e^{-ix} \\ ie^{ix} & -ie^{-ix} \end{vmatrix} = -i - i = -2i(\neq 0). \tag{22}$$

is evaluated to be non-vanishing. Therefore, we can conclude that

$$c_0^* = c_1, \tag{23}$$

or

$$\begin{aligned} x(t) &= c_0 e^{i\omega t} + c_0^* e^{-i\omega t} \\ &= (c_0 + c_0^*)\cos(\omega t) + i\,(c_0 - c_0^*)\sin(\omega t) \\ &= 2\mathrm{Re}(c_0)\cos(\omega t) - 2\mathrm{Im}(c_0)\sin(\omega t) \end{aligned} \tag{24}$$

If we express $\mathcal{C}_0 = 2\mathrm{Re}(c_0), \mathcal{C}_1 = -2\mathrm{Im}(c_0)$ ($\mathcal{C}_k$ are both real constants), then $x(t)$ is nothing but a linear combination of the trigonometric functions, as we already saw in the first lecture (on April 5, 2017).

This result also reminds us of a mathematical nature of a complex number $z = x + iy$, which carries information of two "effective" degrees of freedom (x, y).

In summary, only through the linear combination, Eq.(11), one can construct a consistent solution with the previously obtained solutions, which are expressed by means of the trigonometric functions and the McLaurin series expansion.