Isotropic two-dimensional Harmonic Oscillator

For Classical Mechanics A, Univ.Tokyo (2017)

Descriptions: The "Komaba" solution is presented for the two-dimensional harmonic oscillator in the polar coordinate representation, in contrast to the "Pasadena" solution presented by Dr. S. Golwala (CalTech).

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1 Isotropic two-dimensional harmonic oscillator in the polar coordinates

The equation of motion for the isotropic two-dimensional harmonic oscillator is given as

$$m\frac{d^2}{dt^2}\mathbf{r}(t) = -m\omega^2\mathbf{r}(t), \tag{1}$$

where the position vector $\mathbf{r}(t)$ is given as

$$\mathbf{r}(t) = r(t)\begin{pmatrix} \cos\theta(t) \\ \sin\theta(t) \end{pmatrix} \equiv r(t)\boldsymbol{e}_r. \tag{2}$$

The unit vector $\boldsymbol{e}_r$ is often referred to as the radial basis vector in the two-dimensional poloar coordinates. The other basis $\boldsymbol{e}_\theta$ is defined to be orthogonal to $\boldsymbol{e}_r$. The usual definition reads

$$\boldsymbol{e}_\theta = \begin{pmatrix} -\sin\theta(t) \\ \cos\theta(t) \end{pmatrix}. \tag{3}$$

The left-hand side of the equation of motion, i.e., the second-order derivative of the position vector, is expressed in the poloar coordinates as

$$\frac{d^2}{dt^2}\mathbf{r}(t) = \left(\ddot{r} - r\dot{\theta}^2\right)\boldsymbol{e}_r + \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)\boldsymbol{e}_\theta. \tag{4}$$

The equation of motion thus gives the radial equation and the angular momentum conservation

$$\ddot{r} - r\dot{\theta}^2 = -\omega^2 r, \tag{5}$$

$$mr^2\dot{\theta} = l(= \text{const.}) \tag{6}$$

Eliminating $\dot{\theta}$ thanks to the angular momentum conservation, the radial equation becomes

$$\ddot{r} = -\omega^2 r + \frac{l^2}{m^2 r^3}. \tag{7}$$

2 Energy integral

Multiplying $\dot{r} = \frac{dr}{dt}$ to the both sides in the above equation, we have

$$\frac{d}{dt}\left(\frac{\dot{r}^2}{2} + \frac{\omega^2 r^2}{2} + \frac{l^2}{2m^2 r^4}\right) = 0. \tag{8}$$

This equation corresponds to the energy conservation, and we have

$$\frac{\dot{r}^2}{2} + \frac{\omega^2 r^2}{2} + \frac{l^2}{2m^2 r^2} = E(= \text{const.}) \tag{9}$$

3 The potential energy

Regarding the third term in the energy conservation as the centrifugal energy, we can consider the sum of the second and the third term as the "potential energy".

$$U(r) = \frac{\omega^2 r^2}{2} + \frac{l^2}{2m^2r^2}. \tag{10}$$

The local minimum $dU/dr = 0$ occurs at

$$r_0 = \sqrt{\frac{l}{m\omega}}, \tag{11}$$

and the value of the local minimum is evaluated to be

$$U(r_0) = \frac{l\omega}{m}. \tag{12}$$

When $E = \frac{l\omega}{m}$, the "kinetic energy" $m\dot{r}^2/2$ must be zero due to the energy conservation. Then, the corresponding motion of the oscillator is circular with its radius being r_0. The rotational frequency is calculated through the angular momentum conservation,

$$\omega_0 \equiv \dot{\theta} = \frac{l}{mr_0^2} = \omega. \tag{13}$$

4 Scaling of dynamicl variables

Let us introduce the scaling of dynamical variables to simplify the equation of motion. First of all, the radial coordinate r is scaled to be

$$r = r_0\rho = \sqrt{\frac{l}{m\omega}}\rho, \tag{14}$$

where ρ is a unitless dynamical variable to represent the radial coordinate. We may call r_0 "the oscillator length", as an analogy to the quantum theory of the harmonic oscillator.
Next, the natural time scale can be given in connection to the oscillator frequency ω. From the dimension analysis, we may write

$$t = t_0\tau = \frac{2\pi}{\omega}\tau. \tag{15}$$

With these scaling, the energy conservation reads

$$\left(\frac{r_0}{t_0}\right)^2 \frac{1}{2}\left(\frac{d\rho}{d\tau}\right)^2 + \omega^2 r_0^2 \frac{\rho^2}{2} + \frac{l^2}{2m^2r_0^2}\frac{1}{\rho^2} = E. \tag{16}$$

or

$$\left(\frac{1}{2\pi}\frac{d\rho}{d\tau}\right)^2 + \rho^2 + \frac{1}{\rho^2} = \mathcal{E}, \tag{17}$$

where the energy is scaled as

$$E = U(r_0)\mathcal{E}. \tag{18}$$

5 Second integral

Now, we are going to attempt the second integral to obain $r(t)$. Firstly, we rewrite the scaled energy conservation equation to

$$\frac{d\rho}{d\tau} = \pm 2\pi\sqrt{\mathcal{E} - \rho^2 - \frac{1}{\rho^2}}. \tag{19}$$

In the literature (including Goldstein), it is recommended to convert the time-derivative to θ-derivative with help of the angular momentum conservation,

$$\frac{dr}{dt} = \frac{dr}{d\theta}\dot{\theta} = \frac{l}{mr^2}\frac{dr}{d\theta}. \tag{20}$$

together with the variable conversion $u = 1/r$. Then, we can obtain the formal expression of the second integral for the potential $V = ar^{n+1}$,

$$\theta = \theta_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{l^2} - \frac{2ma}{l^2}u^{-n-1} - u^2}}. \tag{21}$$

According to Goldstein, we can solve this integral analytically for $n = 1, -2, -3$ in terms of the trigonometric expressions, and for $n = 5, 3, 0, -4, -5, -7$ in terms of the (Jacobi's) ellipitic functions. The ellipitic solutions are out of our interest in the current consideration, so that we concentrate on the cases $n = 1, -2, -3$. The harmonic oscillator corresponds to $n = 1$, while the gravity (the inverse-square law) and the centrifugal force correspond to $n = -2$ and $n = -3$, respectively. The θ-derivative has an advantage for the negative-power cases, but the harmonic oscillator receives no benefit for this variable conversion, as demonstrated in the lecture (by coincidence ...). Therefoe, we directly carry out the time integral in the following.

For the harmonic oscillator, we have

$$\pm 2\pi(\tau - \tau_0) = \int_{\rho_0}^{\rho} \frac{d\rho}{\sqrt{\mathcal{E} - \rho^2 - \rho^{-2}}}. \tag{22}$$

To remove the square root, we introduce $\rho = \sqrt{\mathcal{E}}\cos\varphi$. The left-hand side of the above equation then turns to

$$\int_{\rho_0}^{\rho} \frac{d\rho}{\sqrt{\mathcal{E} - \rho^2 - \rho^{-2}}} = -\int_{\varphi_0}^{\varphi} \frac{\sin\varphi\, d\varphi}{\sqrt{\sin^2\varphi - \frac{1}{\mathcal{E}^2\cos^2\varphi}}}. \tag{23}$$

Due to an identity

$$\sin^2\varphi - \frac{1}{\mathcal{E}^2\cos^2\varphi} = \frac{\mathcal{E}^2\cos^2\varphi\sin^2\varphi - 1}{\mathcal{E}^2\cos^2\varphi} = \frac{\sin^2(2\varphi) - 4/\mathcal{E}^2}{4\cos^2\varphi} = \frac{\left(1 - 4/\mathcal{E}^2\right) - \cos^2(2\varphi)}{4\cos^2\varphi}, \tag{24}$$

the integral can be rewritten

$$-\int_{\varphi_0}^{\varphi} \frac{2\sin\varphi\cos\varphi\, d\varphi}{\sqrt{(1 - 4/\mathcal{E}^2) - \cos^2(2\varphi)}} = -\frac{1}{\sqrt{(1 - 4/\mathcal{E}^2)}}\int_{\varphi_0}^{\varphi} \frac{\sin(2\varphi)}{\sqrt{1 - \frac{\cos^2(2\varphi)}{1 - 4/\mathcal{E}^2}}}d\varphi. \tag{25}$$

This is the well-known integral to define the arccos function. By putting $\cos\chi = \cos(2\varphi)/\sqrt{1 - 4/\mathcal{E}^2}$, we have the solution of the integral as $(-\chi + \chi_0)/2$. That is, the result of the second integral reduces to

$$\pm 4\pi(\tau - \tau_0) = -\chi + \chi_0, \tag{26}$$

or

$$\cos\chi = \cos\left(4\pi\tau + \delta_0\right). \tag{27}$$

Here, the phase δ_0 is defined as the sum involving the constants τ_0 and χ_0. Putting back the above result to the expression with ρ, we have

$$\cos(2\varphi) = 2\cos^2\varphi - 1 = \sqrt{1 - 4/\mathcal{E}^2}\cos(4\pi\tau + \delta_0), \tag{28}$$

or

$$\rho^2 = \sqrt{\mathcal{E}^2/4 - 1}\cos(4\pi\tau + \delta_0) + \mathcal{E}/2 \tag{29}$$

Finally, we can scale back to the original dynamical variables.

$$r(t) = r_0\sqrt{\mathcal{A}\cos(2\omega t + \delta_0) + \mathcal{B}}, \tag{30}$$

where constats $\mathcal{A}, \mathcal{B}$ are given as

$$\mathcal{A} = \sqrt{(\mathcal{B}-1)(\mathcal{B}+1)}, \quad \mathcal{B} = \mathcal{E}/2, \quad \mathcal{E} \geq 2. \tag{31}$$

Let us call our solution the "Komaba solution", in comparison to the "Pasadena solution" presented by Dr. S. Golwala. It should be interesting to examine why the energy range is restricted to $\mathcal{E} \geq 2$, which is left to students for analysis.

6 Meaning of the doubling in rotational frequency

The above solution gives rise to the elliptic trajectory, which coincides with the trajectory produced by the gravity (the inverse-square law). However, the frequency in the Komaba solution is doubled in comparison to the original frequency ω. The reason for this doubling should be understood through a physical consideration, which was actually discussed in the lecture. We will come back to this problem again in the gravity section, where the elliptic trajectory is given in a different mathematical expression,

$$r(t) = \frac{a(1-\epsilon^2)}{1+\epsilon\cos(\theta+\delta)}, \tag{32}$$

where $a = -\frac{GMm}{2E}$ is the semi-major axis and ϵ denotes the eccentricity

$$\epsilon = \sqrt{1 + \frac{2l^2E}{m^3G^2M^2}}. \tag{33}$$