Vector potential of the basis in the spherical coordinates

For Electromagnetism A, Univ.Tokyo (2016)

Descriptions: Correction to the last question in QI, in the 2016 final exam, which is about the vector potential for the basis in the spherical coordinates

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1 Divergence of the spherical-coordinate basis

In the three-dimensional spherical coordinates, the basis are given

$$\boldsymbol{e}_r = \begin{pmatrix} \cos\phi\sin\theta \\ \sin\phi\sin\theta \\ \cos\theta \end{pmatrix}, \quad \boldsymbol{e}_\theta = \begin{pmatrix} \cos\phi\cos\theta \\ \sin\phi\cos\theta \\ -\sin\theta \end{pmatrix}, \quad \boldsymbol{e}_\phi = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}. \tag{1}$$

Only the last basis vector, that is, $\boldsymbol{e}_\phi$, gives the vanishing divergence,

$$\nabla\cdot\boldsymbol{e}_\phi = 0. \tag{2}$$

According to a vector identity, $div\ curl = 0$, as seen in the Gauss law for the magnetic field, the basis vector $\boldsymbol{e}_\phi$ can be expressed as $\boldsymbol{e}_\phi = \nabla\times\boldsymbol{A}$. Let us find this vector-potential field $\boldsymbol{A}$ in below.

2 Expression of the derivative operator ∇ in the spherical coordinates

Let us present the expression first.

$$\nabla = \boldsymbol{e}_r\frac{\partial}{\partial r} + \boldsymbol{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \boldsymbol{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}. \tag{3}$$

We then express the vector potential as

$$\boldsymbol{A}(\boldsymbol{r}) = A_r\boldsymbol{e}_r + A_\theta\boldsymbol{e}_\theta + A_\phi\boldsymbol{e}_\phi. \tag{4}$$

Finally, we need to perform the following

$$\nabla\times\boldsymbol{A} = \left(\boldsymbol{e}_r\frac{\partial}{\partial r} + \boldsymbol{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \boldsymbol{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\times(A_r\boldsymbol{e}_r + A_\theta\boldsymbol{e}_\theta + A_\phi\boldsymbol{e}_\phi) \tag{5}$$

To carry out this calculation, we may make use of the following fomrula,

$$(\boldsymbol{A}\partial)\times(f\boldsymbol{B}) = (\partial f)(\boldsymbol{A}\times\boldsymbol{B}) + f(\boldsymbol{A}\times\partial\boldsymbol{B}), \tag{6}$$

which has been derived when we considered the expression of curls in the cylindrical coordinates.

3 Curl in the spherical coordinates

The obtained result reads

$$\begin{aligned} \nabla\times\boldsymbol{A} &= \boldsymbol{e}_r\left(\frac{1}{r}\frac{\partial A_\phi}{\partial\theta} - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\phi} + A_\phi\frac{\cos\theta}{r\sin\theta}\right) + \boldsymbol{e}_\theta\left(-\frac{\partial A_\phi}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{A_\phi}{r}\right) \\ &+ \boldsymbol{e}_\phi\left(\frac{\partial A_\theta}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial\theta} + \frac{A_\theta}{r}\right). \end{aligned} \tag{7}$$

The condition we reqire is

$$\boldsymbol{e}_\phi = \boldsymbol{\nabla} \times \mathbf{A}, \tag{8}$$

so that we need to demand

$$\begin{aligned} \frac{1}{r}\frac{\partial A_\phi}{\partial \theta} - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial \phi} + A_\phi \frac{\cos\theta}{r\sin\theta} &= 0, & (9)\\ -\frac{\partial A_\phi}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{A_\phi}{r} &= 0, & (10)\\ \frac{\partial A_\theta}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \theta} + \frac{A_\theta}{r} &= 1. & (11) \end{aligned}$$

to be satisfied.

4 A special case: $\boldsymbol{A}(\mathbf{r}) = A_r(\mathbf{r})\boldsymbol{e}_r$

In the exam, I asked to find a special solution for the equations above, which is the case when the vector potential is parallel to the radial basis $\boldsymbol{e}_r$. That is,

$$\mathbf{A}(\mathbf{r}) = A_r(r, \theta, \phi)\boldsymbol{e}_r, \tag{12}$$

which also means $A_\theta = A_\phi = 0$. The first equation of the three differential equations above is satisfied in a trival way. The second and third equations turn to

$$\begin{aligned} \frac{\partial A_r}{\partial \phi} &= 0, & (13)\\ \frac{\partial A_r}{\partial \theta} &= -r. & (14) \end{aligned}$$

The first equation means that A_r does not depend on variable ϕ, so that $A_r = A_r(r, \theta)$. Inserting this result to the second equation gives rise to

$$A_r(r, \theta) = -r\theta + X(r), \tag{15}$$

where $X(r)$ is an arbitrary function of r.
Hence,

$$\boldsymbol{e}_\phi = \boldsymbol{\nabla} \times \{(-r\theta + X(r))\,\boldsymbol{e}_r\}. \tag{16}$$